Lecture 2:

Recall:

1. Real world problems can be formulated into mathematical equations (usually involves derivatives);
2. Our main focus in Math 3310:

a. How to solve equations analytically and numerically;
b. Analyse the numerical algorithm (convergence to solution?)
c. Analyse the numerical approximation (accuracy?)

Analytic methods for solving differential equation

Note: Most differential equations do not have analytic (exact) solutions!

e.g. $$-\frac{d}{dx}\Big(\underbrace{\overset{\overset{x}{\wr\wr}}{c(x)}}_{\text{complicated}}\,u(x)\Big) = \underbrace{\overset{\overset{x^2}{\wr\wr}}{f(x)}}_{\text{complicated}}$$

DOESN'T have analytic sol!

$\downarrow$ convert / approximation

$$-\frac{d}{dx}\big(x\,u(x)\big) = x^2$$

(Have analytic solution! Give a good initial guess for the sol. of the original equation!)

Three most basic techniques:

(1) Integrating factor

(2) Separation of variables

(3) Analytic spectral (Fourier) method

(1) Integrating factor

(A) First order differential equation (involving first derivatives ONLY)

Consider: (★) $\dfrac{dy}{dx} + P(x)\, y(x) = Q(x)$ (y is unknown function)

Let $M(x) = e^{\int_{s_0}^{x} P(s)\,ds}$ ← some constant. Then, it is easy to check:

$$\frac{d}{dx}\big(M(x)\,y(x)\big) = \frac{dM(x)}{dx}\, y(x) + M(x)\frac{dy}{dx}$$

$$= \underbrace{e^{\int_{s_0}^{x} P(s)\,ds}}_{M(x)} P(x)\, y(x) + M(x)\frac{dy}{dx}$$

$$\therefore \frac{d}{dx}\big(M(x)\,y(x)\big) = M(x)\left(\frac{dy}{dx} + P(x)\, y(x)\right)$$

Multiply both sides of $(*)$ by $M(x)$:

$$M(x)\underbrace{\left(\frac{dy}{dx} + P(x)y(x)\right)}_{\frac{d}{dx}\left(M(x)y(x)\right)} = M(x)\,Q(x)$$

$$\therefore \quad \int \frac{d}{dx}\left(M(x)y(x)\right) = \int M(x)\,Q(x)$$

$$\Rightarrow \quad M(x)y(x) = \int M(x)\,Q(x)\,dx + C$$

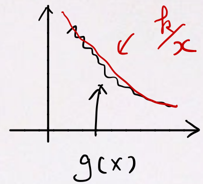
$$y(x) = \left[\int \left(e^{\int_{s_0}^{x} P(s)\,ds}\right) Q(x)\,dx + C\right]\left(e^{-\int_{s_0}^{x} P(s)\,ds}\right)$$

Remark: $M(x)$ is called the <u>integrating factor</u>.

Example 1: Consider : $\dfrac{dy}{dx} - g(x)\,y(x) = 0$, $1 \le x < \infty$ with $y(1) = 1$.

Suppose $g(x) \approx \dfrac{k}{x}$.

Find an approximated guess of $y(x)$.



Solution: Consider : $\dfrac{dy}{dx} - \dfrac{k}{x}\,y(x) = 0$

Let $M(x) = e^{\int -\frac{k}{x}\,dx} = e^{-k\ln x} = x^{-k}$

Then: $M(x)\left(\dfrac{dy}{dx} - \dfrac{k}{x}\,y(x)\right) = 0 \cdot M(x)$

$$\Rightarrow \frac{d}{dx}\big(M(x)\,y(x)\big) = 0$$

$$\Rightarrow x^{-k} \approx M(x)\,y(x) = C \Rightarrow y(x) = Cx^{k}.$$

$\because$ $y(1) = 1 \Rightarrow 1 = C$. $\therefore$ $y(x) = x^k$ is an approximated guess of the solution.

Example 2: Consider:

$$f(x)\frac{dy}{dx} + g(x)\,y(x) = h(x), \quad 2 \le x < \infty \text{ with } y(2) = 1$$

Suppose $f(x) \approx (x^2-1)$; $g(x) \approx 2x$; $h(x) \approx x$.

Find an approximated guess of $y(x)$.

Solution: Consider: $\dfrac{dy}{dx} + \dfrac{2x}{x^2-1}\,y(x) = \dfrac{x}{x^2-1}$

Let $M(x) = e^{\int \frac{2x}{x^2-1}dx} = e^{\ln(x^2-1)} = x^2-1$

Then: $M(x)\left(\dfrac{dy}{dx} + \dfrac{2x}{x^2-1}\,y(x)\right) = M(x)\left(\dfrac{x}{x^2-1}\right)$

$$\Rightarrow \frac{d}{dx}\left(M(x)\,y(x)\right) = M(x)\frac{x}{x^2-1}$$

$$\therefore \int \frac{d}{dx}\left((x^2-1)\,y(x)\right) = \int \cancel{(x^2-1)}\left(\frac{x}{\cancel{x^2-1}}\right)$$

$$\Rightarrow \quad (x^2-1)\,y(x) = \frac{x^2}{2} + C$$

$$\Rightarrow \quad y(x) = \left(\tfrac{1}{2}x^2 + C\right) / (x^2-1)$$

$$y(2) = 1 \Rightarrow \quad 1 = (C+2)/3 \Rightarrow C = 1.$$

$\therefore\ y(x) = \left(\tfrac{1}{2}x^2 + 1\right)/(x^2-1)$ is an approximated guess of the solution.

(B) Second order differential equation (involving second derivatives)

Consider : $-c\dfrac{d^2u}{dx^2} + q\,u(x) = 0$ where $c>0$, $q>0$ are positive constants.

Let $M(x) = \dfrac{du}{dx}$ (integrating factor)

Then: $$c\frac{d^2u}{dx^2}\underbrace{M(x)}_{=\frac{du}{dx}} = q\,u(x)\underbrace{M(x)}_{=\frac{du}{dx}}$$

$$\Leftarrow \quad \frac{d}{dx}\left(c\left(\frac{du}{dx}\right)^2\right) = \frac{d}{dx}\left(q\,(u(x))^2\right)$$

A possible solution of the above is :

$$c\left(\frac{du}{dx}\right)^2 = q\,(u(x))^2$$

$$\therefore \quad \frac{du}{dx} = \pm\sqrt{\frac{q}{c}}\,u(x)$$

Using the integrating factor technique for 1st order differential eqt:

$$u(x) = K e^{\pm\sqrt{\frac{q}{c}}\,x} \quad \text{for some constant } K.$$

For general solution, $u(x) = \alpha_1 e^{\sqrt{\frac{q}{c}}\,x} + \alpha_2 e^{-\sqrt{\frac{q}{c}}\,x}$

where α_1 and α_2 are some constants determined by boundary conditions.

<u>Example</u>: Assume $u(0) = 0$ and $u(1) = 2$.

We get
$$\alpha_1 + \alpha_2 = 0$$
$$\alpha_1 e^{\sqrt{\frac{q}{c}}} + \alpha_2 e^{-\sqrt{\frac{q}{c}}} = 2$$

$$\Rightarrow \alpha_1 = -\alpha_2 = \frac{2}{e^{\sqrt{\frac{q}{c}}} - e^{-\sqrt{\frac{q}{c}}}}$$

Example: (Non-homogeneous case)

Consider:

$$(*)\begin{cases} -c\dfrac{d^2u}{dx^2} + q\,u = q\,x^2 + 1 \\ \dfrac{du}{dx}(0) = 1, \quad u(1) = 1 \end{cases}$$

Note that if $w(x)$ satisfies $(*)$, then:

$$u(x) = \underbrace{\alpha_1 e^{\sqrt{\frac{q}{c}}x} + \alpha_2 e^{-\sqrt{\frac{q}{c}}x}}_{\text{Homogeneous}} + \underbrace{w(x)}$$ for some constants

α_1 and α_2 is a general sol.

In our case, $w(x) = x^2 + \left(\dfrac{2c+1}{q}\right)$ is a solution.

$$\therefore\ u(x) = \alpha_1 e^{\sqrt{\frac{q}{c}}x} + \alpha_2 e^{-\sqrt{\frac{q}{c}}x} + x^2 + \left(\frac{2c+1}{q}\right)$$

determined by boundary conditions.

Another useful technique: Separation of variables

Consider a heat equation (on a unit circle):

$$u_t = u_{xx}, \quad x \in [0, 2\pi], \quad t \geq 0$$

subject to: $\begin{cases} u(0,t) = u(2\pi, t) & \text{(periodic condition)} \\ u(x,0) = \sin x & \text{(initial condition)} \end{cases}$

Strategy: Let $u(x,t) = X(x)\,T(t)$.

$$u_t = u_{xx} \Rightarrow X(x)\,T'(t) = X''(x)\,T(t)$$

$$\therefore \frac{X''}{X} = \frac{T'}{T} = \lambda \leftarrow \text{some constant}$$

Then: we get two ordinary differential equations:

$$X''(x) = \lambda X(x) \quad \text{and} \quad T'(t) = \lambda T(t)$$

(Partial differential eqt $\Rightarrow$ 2 Ordinary differential eqt)